

ON THE IMAGE OF THE ASSOCIATED FORM MORPHISM

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ABSTRACT. Let $\mathbb{C}[x_1, \dots, x_n]_{d+1}$ be the vector space of homogeneous forms of degree $d+1$ on \mathbb{C}^n , with $n, d \geq 2$. In earlier articles by J. Alper, M. Eastwood and the author, we introduced a morphism, called A , that assigns to every nondegenerate form the so-called associated form lying in the space $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$. One of the reasons for our interest in A is the conjecture—motivated by the well-known Mather-Yau theorem on complex isolated hypersurface singularities—asserting that all regular GL_n -invariant functions on the affine open subvariety $\mathbb{C}[x_1, \dots, x_n]_{d+1, \Delta}$ of forms with nonvanishing discriminant can be obtained as the pull-backs by means of A of the rational GL_n -invariant functions on $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ defined on $\mathrm{im}(A)$. The morphism A factors as $A = \mathbf{A} \circ \mathrm{grad}$, where grad is the gradient morphism and \mathbf{A} assigns to every n -tuple of forms of degree d with nonvanishing resultant a form in $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ defined analogously to $A(f)$ for a nondegenerate f . In order to establish the conjecture, it is important to study the image of \mathbf{A} . In the present paper, we show that $\mathrm{im}(\mathbf{A})$ is an open subset of an irreducible component of each of the so-called catalecticant varieties V , $\mathrm{Gor}(T)$ and describe the closed complement to $\mathrm{im}(\mathbf{A})$, at the same time clarifying and extending known results on these varieties. Furthermore, for $n = 3$, $d = 2$ we give a description of the complement to $\mathrm{im}(\mathbf{A})$ via the zero locus of the Aronhold invariant of degree 4, which establishes an analogy with the case $n = 2$ where this complement is known to be the vanishing locus of the catalecticant for any $d \geq 2$.

1. INTRODUCTION

This paper is motivated by a new construction in classical invariant theory that originated in article [EI] and was further explored in [AI1], [AI2], [AIK]. Fix integers $n \geq 2$ and $d \geq 2$ and let $\mathbb{C}[x_1, \dots, x_n]_{d+1, \Delta}$ be the complex affine open subvariety of the space $\mathbb{C}[x_1, \dots, x_n]_{d+1}$ of homogeneous forms of degree $d+1$ in n variables where the discriminant Δ does not vanish. Consider the Milnor algebra $M(f) := \mathbb{C}[x_1, \dots, x_n]/(f_{x_1}, \dots, f_{x_n})$ of the isolated singularity at the origin of the hypersurface in \mathbb{C}^n defined by $f \in \mathbb{C}[x_1, \dots, x_n]_{d+1, \Delta}$ and let $\mathfrak{m} \subset M(f)$ be the maximal ideal. One can then introduce a form on the n -dimensional quotient $\mathfrak{m}/\mathfrak{m}^2$ with values in the one-dimensional socle $\mathrm{Soc}(M(f))$ of $M(f)$ as follows:

$$\mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathrm{Soc}(M(f)), \quad z \mapsto \widehat{z}^{n(d-1)},$$

where \widehat{z} is any element of \mathfrak{m} that projects to $z \in \mathfrak{m}/\mathfrak{m}^2$. There is a canonical isomorphism $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{C}^n$ and, since the Hessian of f generates the socle, there is also a canonical isomorphism $\mathrm{Soc}(M(f)) \cong \mathbb{C}$. Hence, one obtains a form $A(f)$ of degree $n(d-1)$ on \mathbb{C}^n , which is called the *associated form of f* . This form is very natural; in particular, it is a Macaulay inverse system for the Milnor algebra $M(f)$.

The main object of our study in [AI1], [AI2], [AIK] was the morphism

$$A: \mathbb{C}[x_1, \dots, x_n]_{d+1, \Delta} \rightarrow \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}, \quad f \mapsto A(f)$$

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of affine varieties. As first observed in [EI], for certain values of n and d one can recover all GL_n -invariant rational functions on forms of degree $d + 1$ from those on forms of degree $n(d - 1)$ by evaluating the latter on associated forms, i.e., by composing them with A . Motivated by the above fact, in [AI1] we proposed a conjecture asserting that an analogous statement holds for all n and d (cf. [EI, Conjecture 3.2]):

Conjecture 1.1. *For any regular GL_n -invariant function J on $\mathbb{C}[x_1, \dots, x_n]_{d+1, \Delta}$ there exists a rational GL_n -invariant function \tilde{J} on $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ defined on the image of A such that $J = \tilde{J} \circ A$.*

In other words, the conjecture asserts that the invariant theory of forms of degree $d + 1$ can be extracted, in a canonical way, from that of forms of degree $n(d - 1)$ at least at the level of rational invariant functions. While this statement is quite intriguing from the purely invariant-theoretic viewpoint, it was originally motivated—as explained in [EI], [AI1]—by complex singularity theory, specifically, by the well-known Mather-Yau theorem (see [MY] and also [Be], [Sh], [GLS, Theorem 2.26]). In [AI2], Conjecture 1.1 was shown to hold for binary forms of any degree, and in [AI1] its weaker variant was established for arbitrary n, d .

In this paper, we obtain results towards settling the conjecture in full generality, which are at the same time of interest in a broader algebraic context. The morphism A factors as $A = \mathbf{A} \circ \mathrm{grad}$, where $\mathrm{grad} : \mathbb{C}[x_1, \dots, x_n]_{d+1} \rightarrow \mathbb{C}[x_1, \dots, x_n]_d^{\oplus n}$ is the gradient morphism and $\mathbf{A} : (\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n})_{\mathrm{Res}} \rightarrow \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ assigns to every n -tuple $\mathbf{f} = (f_1, \dots, f_n)$ of forms of degree d with nonvanishing resultant the *associated form of \mathbf{f}* defined analogously to $A(f)$, with the partial derivative f_{x_j} replaced by f_j for all j . Note that for every \mathbf{f} the form $\mathbf{A}(\mathbf{f})$ is a Macaulay inverse system for the zero-dimensional complete intersection algebra $M(\mathbf{f}) := \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$. As explained in [AI2, Section 3], in order to establish Conjecture 1.1 for all n, d , it is important to study the image of \mathbf{A} . In this paper we show that $\mathrm{im}(\mathbf{A}) \subset \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ is an open subset of an irreducible component of each of the *catalecticant varieties* V and $\mathrm{Gor}(T)$ (see Section 3 for the definitions) and give a description of the closed complement to $\mathrm{im}(\mathbf{A})$. We note that a number of other properties of the morphism \mathbf{A} (as well as the gradient morphism) essential for confirming Conjecture 1.1 were obtained in the recent paper [F].

The irreducible components of catalecticant varieties are of general interest and have been studied regardless of Conjecture 1.1 (see [IK, Chapter 4] and references therein for details). In particular, in [IK, Theorem 4.17] it was shown that $\mathrm{Gor}(T)$ has an irreducible component containing $\mathrm{im}(\mathbf{A})$ as a dense subset and the dimension of this component was found. On the other hand, an analogous fact for V (which is the catalecticant variety most relevant to our study of the morphism \mathbf{A}) appears to be only known in the cases (i) $n = 3, d \geq 3$, (ii) $n = 4, d = 2, 3$, (iii) $n = 5, d = 2$ (see [IK, Theorem 4.19 and Corollary 4.18]), and one of our aims is to bring the results on V in line with those on $\mathrm{Gor}(T)$.

In this paper, we, first of all, refine and extend Theorems 4.17 and 4.19 of [IK]. Namely, in Section 3 we show that the set $\mathrm{im}(\mathbf{A})$ is open (not just dense) in an irreducible component of each of $V, \mathrm{Gor}(T)$ for all n, d and explicitly describe the closed complement to $\mathrm{im}(\mathbf{A})$ (see Theorem 3.3). Note that finding a suitable characterization of this complement is important for resolving Conjecture 1.1 (see Remark 3.5). As the proof of Theorem 4.17 in [IK] is quite brief, we also provide an alternative derivation—with full details—of the dimension formula for $\mathrm{im}(\mathbf{A})$. Note that, although we assume the base field to be \mathbb{C} , our arguments work for any algebraically closed field k of characteristic zero and even apply to

the case $\text{char}(k) > n(d-1)$, with $n(d-1)$ being the socle degree of $M(\mathbf{f})$ for all $\mathbf{f} = (f_1, \dots, f_n) \in (k[x_1, \dots, x_n]_d^{\oplus n})_{\text{Res}}$. We also stress that our clarifications and extensions of results of [IK] only apply in the case of zero-dimensional complete intersections with homogeneous ideal generators of equal degrees.

In fact, ideally, one would like to have a better description of the complement to $\text{im}(\mathbf{A})$ than the one provided by Theorem 3.3. Namely, it would be desirable to represent it as the intersection of the relevant irreducible component of V with the zero locus of an SL_n -invariant form on $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$. This is indeed possible for $n = 2$, in which case the SL_2 -invariant in question is the catalecticant (see [AI2, Proposition 4.3]). In Section 4 we show that such a representation is also valid for $n = 3$, $d = 2$, with the corresponding SL_3 -invariant being the Aronhold invariant of degree 4 (see Proposition 4.1).

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2. PRELIMINARIES ON ASSOCIATED FORMS AND THE MORPHISM \mathbf{A}

In this section we introduce the main object of our study. What follows is an abridged version of the exposition given in [AI2, Section 2].

Fix $n \geq 2$ and for any nonnegative integer j define $\mathbb{C}[x_1, \dots, x_n]_j$ to be the vector space of homogeneous forms of degree j in x_1, \dots, x_n over \mathbb{C} . Clearly, one has $\mathbb{C}[x_1, \dots, x_n] = \bigoplus_{j=0}^{\infty} \mathbb{C}[x_1, \dots, x_n]_j$. Next, fix $d \geq 2$ and consider the vector space $\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n}$ of n -tuples $\mathbf{f} = (f_1, \dots, f_n)$ of forms of degree d . Recall that the resultant Res on the space $\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n}$ is a form with the property that $\text{Res}(\mathbf{f}) \neq 0$ if and only if f_1, \dots, f_n have no common zeroes away from the origin (see, e.g., [GKZ, Chapter 13]).

For $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{C}[x_1, \dots, x_n]_d^{\oplus n}$, we now introduce the algebra

$$M(\mathbf{f}) := \mathbb{C}[x_1, \dots, x_n] / (f_1, \dots, f_n)$$

and recall a well-known lemma (see, e.g., [AI2, Lemma 2.4] and [SS, p. 187]):

Lemma 2.1. *The following statements are equivalent:*

- (1) *the resultant $\text{Res}(\mathbf{f})$ is nonzero;*
- (2) *the algebra $M(\mathbf{f})$ has finite vector space dimension;*
- (3) *the morphism $\mathbf{f}: \mathbb{A}^n(\mathbb{C}) \rightarrow \mathbb{A}^n(\mathbb{C})$ is finite;*
- (4) *the n -tuple \mathbf{f} is a homogeneous system of parameters of $\mathbb{C}[x_1, \dots, x_n]$, i.e., the Krull dimension of $M(\mathbf{f})$ is 0.*

If the above conditions are satisfied, then $M(\mathbf{f})$ is a local standard graded complete intersection algebra whose socle $\text{Soc}(M(\mathbf{f}))$ is generated in degree $n(d-1)$ by the image $\overline{\text{jac}(\mathbf{f})} \in M(\mathbf{f})$ of the Jacobian $\text{jac}(\mathbf{f}) := \det \text{Jac}(\mathbf{f})$, where $\text{Jac}(\mathbf{f})$ is the Jacobian matrix $(\partial f_i / \partial x_j)_{i,j}$.

Remark 2.2. As we pointed out in Lemma 2.1, the algebra $M(\mathbf{f})$ has a natural standard grading: $M(\mathbf{f}) = \bigoplus_{i=0}^{\infty} M(\mathbf{f})_i$. It is well-known (see, e.g., [St, Corollary 3.3]) that the corresponding Hilbert function $H(x) := \sum_{i=0}^{\infty} t_i x^i$, with $t_i := \dim_{\mathbb{C}} M(\mathbf{f})_i$, is given by

$$(2.1) \quad H(x) = (x^{d-1} + \dots + x + 1)^n.$$

Next, we let $(\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n})_{\text{Res}}$ be the affine open subvariety of $\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n}$ that consists of all n -tuples of forms with nonzero resultant. We now define the *associated form* $\mathbf{A}(\mathbf{f}) \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ of $\mathbf{f} = (f_1, \dots, f_n) \in (\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n})_{\text{Res}}$ by the formula

$$(y_1 \bar{x}_1 + y_2 \bar{x}_2 + \dots + y_n \bar{x}_n)^{n(d-1)} = \mathbf{A}(\mathbf{f})(y_1, \dots, y_n) \cdot \overline{\text{jac}(\mathbf{f})} \in M(\mathbf{f}),$$

where $\bar{x}_i \in M(\mathbf{f})$ is the image of x_i . It is not hard to see that the induced map

$$\mathbf{A}: (\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n})_{\text{Res}} \rightarrow \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}, \quad \mathbf{f} \mapsto \mathbf{A}(\mathbf{f})$$

is a morphism of affine varieties. This morphism is quite natural; in particular, it has an important equivariance property (see [AI2, Lemma 2.7]). In article [AI2] we studied \mathbf{A} in relation to Conjecture 1.1 stated in the introduction.

We will now interpret \mathbf{A} in different terms. Recall that the algebra $\mathbb{C}[y_1, \dots, y_n]$ is a $\mathbb{C}[x_1, \dots, x_n]$ -module via differentiation:

$$(2.2) \quad (h \diamond F)(y_1, \dots, y_n) := h \left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right) F(y_1, \dots, y_n),$$

where $h \in \mathbb{C}[x_1, \dots, x_n]$ and $F \in \mathbb{C}[y_1, \dots, y_n]$. For a positive integer j , differentiation induces a perfect pairing

$$\mathbb{C}[x_1, \dots, x_n]_j \times \mathbb{C}[y_1, \dots, y_n]_j \rightarrow \mathbb{C}, \quad (h, F) \mapsto h \diamond F;$$

it is often referred to as the *polar pairing*. For $F \in \mathbb{C}[y_1, \dots, y_n]_j$, we now introduce the homogenous ideal, called the *annihilator* of F ,

$$F^\perp := \{h \in \mathbb{C}[x_1, \dots, x_n] \mid h \diamond F = 0\} \subset \mathbb{C}[x_1, \dots, x_n],$$

which is clearly independent of scaling and thus is well-defined for F in the projective space $\mathbb{P}(\mathbb{C}[y_1, \dots, y_n]_j)$. It is well-known that the quotient $\mathbb{C}[x_1, \dots, x_n]/F^\perp$ is a standard graded local Artinian Gorenstein algebra of socle degree j and the following holds (cf. [IK, Lemma 2.12]):

Proposition 2.3. *The correspondence $F \mapsto \mathbb{C}[x_1, \dots, x_n]/F^\perp$ induces a bijection*

$$\mathbb{P}(\mathbb{C}[y_1, \dots, y_n]_j) \rightarrow \left\{ \begin{array}{l} \text{local Artinian Gorenstein algebras } \mathbb{C}[x_1, \dots, x_n]/I \\ \text{of socle degree } j, \text{ where the ideal } I \text{ is homogeneous} \end{array} \right\}.$$

Remark 2.4. Given a homogenous ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ such that $\mathbb{C}[x_1, \dots, x_n]/I$ is a local Artinian Gorenstein algebra of socle degree j , Proposition 2.3 implies that there is a form $F \in \mathbb{C}[y_1, \dots, y_n]_j$, unique up to scaling, such that $I = F^\perp$. In fact, the uniqueness part of this statement can be strengthened: if $I \subset F^\perp$, then $I = F^\perp$ and all forms with this property are mutually proportional. Indeed, $I \subset F^\perp$ implies $I_j \subset F^\perp$, where $I_j := I \cap \mathbb{C}[x_1, \dots, x_n]_j$, and the claim follows from the fact that I_j has codimension 1 in $\mathbb{C}[x_1, \dots, x_n]_j$. Any such form F is called a (*homogeneous*) *Macaulay inverse system* for $\mathbb{C}[x_1, \dots, x_n]/I$ and its image in $\mathbb{P}(\mathbb{C}[y_1, \dots, y_n]_j)$ is called *the (homogeneous) Macaulay inverse system* for $\mathbb{C}[x_1, \dots, x_n]/I$.

We have (see [AI2, Proposition 2.11]):

Proposition 2.5. *For any $\mathbf{f} \in (\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n})_{\text{Res}}$, the form $\mathbf{A}(\mathbf{f})$ is a Macaulay inverse system for the algebra $M(\mathbf{f})$.*

By Proposition 2.5, the morphism \mathbf{A} can be thought of as a map assigning to every element $\mathbf{f} \in (\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n})_{\text{Res}}$ a particular Macaulay inverse system for the algebra $M(\mathbf{f})$.

We now let $U_{\text{Res}} \subset \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ be the locus of forms F such that the subspace $F^\perp \cap \mathbb{C}[x_1, \dots, x_n]_d$ is n -dimensional and has a basis with nonvanishing resultant. It is easy to see that U_{Res} is locally closed in $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$, hence is a variety (see, e.g., Proposition 3.2 below for details). By Proposition 2.5, the image of \mathbf{A} is contained in U_{Res} . Moreover, if $F \in U_{\text{Res}}$, then for the ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ generated by $F^\perp \cap \mathbb{C}[x_1, \dots, x_n]_d$, we have the inclusion $I \subset F^\perp$. By Remark 2.4, the form F is the inverse system for $\mathbb{C}[x_1, \dots, x_n]/I$, and therefore $F = \mathbf{A}(\mathbf{f})$ for some basis $\mathbf{f} = (f_1, \dots, f_n)$ of $F^\perp \cap \mathbb{C}[x_1, \dots, x_n]_d$. Thus, we have proved:

Proposition 2.6. $\text{im}(\mathbf{A}) = U_{\text{Res}}$.

The constructions of the morphism \mathbf{A} can be projectivized. Indeed, denote by $\mathrm{Gr}(n, \mathbb{C}[x_1, \dots, x_n]_d)$ the Grassmannian of n -dimensional subspaces of the space $\mathbb{C}[x_1, \dots, x_n]_d$. The resultant Res on $\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n}$ descends to a section, also denoted by Res , of a power of the very ample generator of the Picard group of $\mathrm{Gr}(n, \mathbb{C}[x_1, \dots, x_n]_d)$. Let $\mathrm{Gr}(n, \mathbb{C}[x_1, \dots, x_n]_d)_{\mathrm{Res}}$ be the affine open subvariety where Res does not vanish; it consists of all n -dimensional subspaces of $\mathbb{C}[x_1, \dots, x_n]_d$ having a basis with nonzero resultant. Consider the morphism

$$(\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n})_{\mathrm{Res}} \rightarrow \mathrm{Gr}(n, \mathbb{C}[x_1, \dots, x_n]_d)_{\mathrm{Res}}, \quad \mathbf{f} = (f_1, \dots, f_n) \mapsto \langle f_1, \dots, f_n \rangle,$$

where $\langle \cdot \rangle$ denotes linear span. Then, by the equivariance property (see [AI2, Lemma 2.7]), the morphism \mathbf{A} composed with the projection $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)} \setminus \{0\} \rightarrow \mathbb{P}(\mathbb{C}[y_1, \dots, y_n]_{n(d-1)})$ factors as

$$(\mathbb{C}[x_1, \dots, x_n]_d^{\oplus n})_{\mathrm{Res}} \rightarrow \mathrm{Gr}(n, \mathbb{C}[x_1, \dots, x_n]_d)_{\mathrm{Res}} \xrightarrow{\hat{\mathbf{A}}} \mathbb{P}(\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}).$$

By Proposition 2.5, the morphism $\hat{\mathbf{A}}$ can be thought of as a map assigning to every subspace $W \in \mathrm{Gr}(n, \mathbb{C}[x_1, \dots, x_n]_d)_{\mathrm{Res}}$ the Macaulay inverse system for the algebra $M(\mathbf{f})$, where $\mathbf{f} = (f_1, \dots, f_n)$ is any basis of W .

Proposition 2.6 implies

Proposition 2.7. $\mathrm{im}(\hat{\mathbf{A}}) = \mathbb{P}(U_{\mathrm{Res}})$, where $\mathbb{P}(U_{\mathrm{Res}})$ is the image of U_{Res} in the projective space $\mathbb{P}(\mathbb{C}[y_1, \dots, y_n]_{n(d-1)})$.

It turns out that $\hat{\mathbf{A}} : \mathrm{Gr}(n, \mathbb{C}[x_1, \dots, x_n]_d)_{\mathrm{Res}} \rightarrow \mathbb{P}(U_{\mathrm{Res}})$ is in fact an isomorphism (see [AI2, Proposition 2.13]). This last result will be utilized in our considerations of the relevant catalecticant varieties in the next section.

3. THE CATALECTICANT VARIETIES AND A DESCRIPTION OF $\mathrm{im}(\mathbf{A})$

Let $K := \dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n]_d = \binom{d+n-1}{n-1}$. Consider the quasiaffine variety $U := U_{K-n}(n(d-1) - d, d; n) \subset \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ and the affine subvariety $V := V_{K-n}(n(d-1) - d, d; n) \subset \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ as defined in [IK, p. 5]. Specifically, set $L := \dim_{\mathbb{C}} \mathbb{C}[y_1, \dots, y_n]_{n(d-1)-d} = \binom{n(d-1)-d+n-1}{n-1}$ and let $\{\mathbf{m}_1, \dots, \mathbf{m}_K\}$, $\{\mathbf{m}_1, \dots, \mathbf{m}_L\}$ be the standard monomial bases in the spaces $\mathbb{C}[x_1, \dots, x_n]_d$ and $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)-d}$, respectively, with the monomials numbered in accordance with some orders, which we will fix from now on. For a form $F \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ let $F_j := \mathbf{m}_j \diamond F \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)-d}$, $j = 1, \dots, K$, where \diamond is defined in (2.2). Expanding F_1, \dots, F_K with respect to $\{\mathbf{m}_1, \dots, \mathbf{m}_L\}$, we obtain an $L \times K$ -matrix $D(F)$ called the *catalecticant matrix*. Then the varieties U and V are described as

$$\begin{aligned} U &= \{F \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)} \mid \mathrm{rank} D(F) = K - n\}, \\ V &= \{F \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)} \mid \mathrm{rank} D(F) \leq K - n\}. \end{aligned}$$

Note that U is a dense open subset of V (see [IK, Lemma 3.5]).

Clearly, $V \subset \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ is the affine subvariety given by the condition of the vanishing of all $(K - n + 1)$ -minors of $D(F)$. Observe that for $n = 2$ one has $K = d + 1$, $L = d - 1$, and therefore the matrix $D(F)$ has no $(K - 1)$ -minors, hence $V = \mathbb{C}[y_1, y_2]_{2(d-1)}$. Similarly, for $n = 3$, $d = 2$, we have $K = 6$, $L = 3$, therefore $D(F)$ has no $(K - 2)$ -minors, hence $V = \mathbb{C}[y_1, y_2, y_3]_3$. Notice that in all other cases one has $L \geq K$, and therefore V is a proper affine subvariety of $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ unless $n = 2$ or $n = 3$, $d = 2$.

Next, let $T := (t_0, t_1, \dots, t_{n(d-1)}) = (1, n, \dots, n, 1)$ be the Gorenstein sequence from the Hilbert function (2.1), which is symmetric about $n(d-1)/2$. Consider the quasiaffine variety $\mathrm{Gor}(T)$ that consists of all forms $F \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ such that the Hilbert function of the standard graded local Artinian Gorenstein algebra

$\mathbb{C}[x_1, \dots, x_n]/F^\perp$ is T . Clearly, $\text{Gor}(T)$ is an open subset of the affine subvariety $\text{Gor}_{\leq}(T) \subset \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ consisting of all forms F for which the Gorenstein sequence of $\mathbb{C}[x_1, \dots, x_n]/F^\perp$ does not exceed T . Analogously to V , the variety $\text{Gor}_{\leq}(T)$ is defined by the vanishing of all $(t_i + 1)$ -minors of the corresponding matrices constructed analogously to $D(F)$, for $i = 1, \dots, n(d-1) - 1$. Following [IK], we call V and $\text{Gor}(T)$ the *catalecticant varieties*.

Remark 3.1. We note that [IK] introduces more general catalecticant varieties (and even schemes), but V and $\text{Gor}(T)$ are the ones most relevant to our study of the morphism \mathbf{A} , thus in the present paper only these two varieties are considered.

We have the obvious inclusions

$$(3.1) \quad U_{\text{Res}} \subset \text{Gor}(T) \subset U \subset V,$$

where $U_{\text{Res}} \subset \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ was defined in Section 2. To better understand the relationship between U_{Res} , $\text{Gor}(T)$, U and V , we will now introduce a certain closed subset of U .

Cover U by open subsets U_α , each of which is given by the condition of the nonvanishing of a particular $(K - n)$ -minor of the catalecticant matrix $D(F)$. In what follows, on each U_α we will define a regular function R_α . Let, for instance, U_{α_0} be the subset of U described by the nonvanishing of the principal $(K - n)$ -minor of $D(F)$. For $F \in U_{\alpha_0}$ we will now find a canonical basis of the solution set $\mathcal{S}(F)$ of the homogeneous system $D(F)\gamma = 0$, where γ is a column-vector in \mathbb{C}^K . Since $\text{rank } D(F) = K - n$, one has $\dim_{\mathbb{C}} \mathcal{S}(F) = n$. Split $D(F)$ into blocks as follows:

$$D(F) = \begin{pmatrix} \boxed{A(F)} & \boxed{B(F)} \\ \boxed{C(F)} \end{pmatrix},$$

where $A(F)$ has size $(K - n) \times (K - n)$ (recall that $\det A(F) \neq 0$), $B(F)$ has size $(K - n) \times n$, and $C(F)$ has size $(L - K + n) \times K$. We also split the column-vector γ as $\gamma = \begin{pmatrix} \gamma' \\ \gamma'' \end{pmatrix}$, where γ' is in \mathbb{C}^{K-n} and γ'' is in \mathbb{C}^n . Then $\mathcal{S}(F)$ is given by the condition $\gamma' = -A(F)^{-1}B(F)\gamma''$. Therefore, the vectors

$$\gamma_j(F) := \begin{pmatrix} -A(F)^{-1}B(F)\mathbf{e}_j \\ \mathbf{e}_j \end{pmatrix}, \quad j = 1, \dots, n,$$

form a basis of $\mathcal{S}(F)$ for every $F \in U_{\alpha_0}$, where \mathbf{e}_j is the j th standard basis vector in \mathbb{C}^n .

Clearly, the components $\gamma_j^1, \dots, \gamma_j^K$ of γ_j are regular functions on U_{α_0} for each j , and we define $r_{j, \alpha_0} := \sum_{i=1}^K \gamma_j^i \mathbf{m}_i$, $j = 1, \dots, n$, where, as before, $\{\mathbf{m}_1, \dots, \mathbf{m}_K\}$ is the standard monomial basis in $\mathbb{C}[x_1, \dots, x_n]_d$. Then the d -forms $r_{1, \alpha_0}(F), \dots, r_{n, \alpha_0}(F)$ constitute a basis of the intersection $F^\perp \cap \mathbb{C}[x_1, \dots, x_n]_d$ for every $F \in U_{\alpha_0}$. Set $R_{\alpha_0} := \text{Res}(r_{1, \alpha_0}, \dots, r_{n, \alpha_0})$. Clearly, R_{α_0} is a regular function on U_{α_0} , and we define Z_{α_0} to be its zero locus.

Arguing as above for every U_α , we introduce a regular function R_α on U_α and its zero locus Z_α . Notice that if for some α, α' the intersection $U_{\alpha, \alpha'} := U_\alpha \cap U_{\alpha'}$ is nonempty, then $Z_\alpha \cap U_{\alpha, \alpha'} = Z_{\alpha'} \cap U_{\alpha, \alpha'}$. Thus, the loci Z_α glue together into a closed subset Z of U . If U' is an irreducible component of U , then the intersection $Z \cap U'$ is either a hypersurface in U' , or all of U' , or empty. Notice also that Z is GL_n -invariant, which follows from the general formula

$$(CF)^\perp \cap \mathbb{C}[x_1, \dots, x_n]_j = C^{-t}(F^\perp \cap \mathbb{C}[x_1, \dots, x_n]_j), \quad j = 0, \dots, n(d-1),$$

for all $C \in \text{GL}_n$, $F \in \mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$.

We will now establish:

Proposition 3.2. *One has $U_{\text{Res}} = \text{Gor}(T) \setminus Z = U \setminus Z = V \setminus \overline{Z}$.*

Proof. It is clear that $U_{\text{Res}} = U \setminus Z$, thus inclusions (3.1) imply $U_{\text{Res}} = \text{Gor}(T) \setminus Z = U \setminus Z$. Further, to see that $U \setminus Z = V \setminus \overline{Z}$, we need to prove that $V \setminus U \subset \overline{Z}$. As shown in the proof of [IK, Lemma 3.5], in every neighborhood of every form $F \in V \setminus U$ there exists $\widehat{F} \in U$ such that all elements of $\widehat{F}^\perp \cap \mathbb{C}[x_1, \dots, x_n]_d$ have a common zero away from the origin. Thus, $F \in \overline{Z}$ as required. \square

Next, by Proposition 2.7, the morphism $\widehat{\mathbf{A}} : \text{Gr}(n, \mathbb{C}[x_1, \dots, x_n]_d)_{\text{Res}} \rightarrow \mathbb{P}(U_{\text{Res}})$ is surjective. In fact, by [AI2, Proposition 2.13], the map $\widehat{\mathbf{A}}$ is an isomorphism, therefore we have

$$\dim_{\mathbb{C}} \mathbb{P}(U_{\text{Res}}) = \dim_{\mathbb{C}} \text{Gr}(n, \mathbb{C}[x_1, \dots, x_n]_d) = Kn - n^2,$$

which implies

$$(3.2) \quad \dim_{\mathbb{C}} U_{\text{Res}} = Kn - n^2 + 1 =: N.$$

As U_{Res} is irreducible, we obtain the following result:

THEOREM 3.3. *There exist irreducible components $\text{Gor}(T)^\circ$, U° , V° of the varieties $\text{Gor}(T)$, U , V , respectively, such that $U_{\text{Res}} = \text{Gor}(T)^\circ \setminus Z = U^\circ \setminus Z = V^\circ \setminus \overline{Z}$, with $\dim_{\mathbb{C}} \text{Gor}(T)^\circ = \dim_{\mathbb{C}} U^\circ = \dim_{\mathbb{C}} V^\circ = N$, where N is defined in (3.2).*

As by Proposition 2.6 we have $\text{im}(\mathbf{A}) = U_{\text{Res}}$, Theorem 2.6 yields a description of the image of the morphism \mathbf{A} in terms of $\text{Gor}(T)$, U , V and Z .

Remark 3.4. Theorem 4.17 of [IK] shows that $\text{Gor}(T)$ has an irreducible component containing U_{Res} as a dense subset and the dimension of this component is equal to N . The proof given in [IK] does not explicitly utilize the morphism \mathbf{A} and is somewhat brief overall. Also, Theorem 4.19 of [IK] (cf. Corollary 4.18 therein) yields that U_{Res} is dense in an irreducible component of V in the following cases: (i) $n = 3$, $d \geq 3$, (ii) $n = 4$, $d = 2, 3$, (iii) $n = 5$, $d = 2$. In comparison with these results, Theorem 3.3 stated above is more precise because:

- it treats both $\text{Gor}(T)$ and V simultaneously for all n, d ;
- it shows that U_{Res} is in fact open (not just dense) in an irreducible component of each of $\text{Gor}(T)$ and V and explicitly describes the closed complement to U_{Res} in terms of the subset Z ;
- its proof gives a complete argument for the formula for $\dim_{\mathbb{C}} U_{\text{Res}}$.

Remark 3.5. Describing the complement to $\text{im}(\mathbf{A}) = U_{\text{Res}}$ in V° is of particular importance for settling Conjecture 1.1. Theorem 3.3 offers a description in terms of the set Z , but, ideally, one would like to show that there exists an SL_n -invariant form on $\mathbb{C}[y_1, \dots, y_n]_{n(d-1)}$ whose zero locus intersects V° in $V^\circ \setminus \text{im}(\mathbf{A})$. This indeed holds for $n = 2$, in which case $V^\circ = V = \mathbb{C}[y_1, y_2]_{2(d-1)}$ and $\mathbb{C}[y_1, y_2]_{2(d-1)} \setminus \text{im}(\mathbf{A})$ is the zero locus of the catalecticant (see [AI2, Proposition 4.3]). The above fact was instrumental for establishing Conjecture 1.1 in the binary case in [AI2]. In the next section we will show that an analogous statement is also valid for $n = 3$, $d = 2$. Notice that, by [EI], the conjecture holds in this situation as well.

4. THE CASE $n = 3$, $d = 2$

In this section we set $n = 3$, $d = 2$. Notice that the associated form of any element of $(\mathbb{C}[x_1, x_2, x_3]_2^{\oplus 3})_{\text{Res}}$ is a ternary cubic and that $V^\circ = V = \mathbb{C}[y_1, y_2, y_3]_3$. Let S be the degree four Aronhold invariant. An explicit formula for S can be found, for example, in [DK, p. 250]. Namely, for a ternary cubic

$$\begin{aligned} c(y_1, y_2, y_3) = & ay_1^3 + by_2^3 + cy_3^3 + 3dy_1^2y_2 + 3ey_1^2y_3 + 3fy_1y_2^2 + \\ & 3gy_2^2y_3 + 3hy_1y_3^2 + 3iy_2y_3^2 + 6jy_1y_2y_3 \end{aligned}$$

one has

$$(4.1) \quad \begin{aligned} S(c) = & abcj - bcde - cavg - abhi - j(agi + bhe + cdf) + \\ & afi^2 + ahg^2 + bdh^2 + bie^2 + cgd^2 + cef^2 - j^4 + \\ & 2j^2(fh + id + eg) - 3j(dgh + efi) - f^2h^2 - i^2d^2 - \\ & e^2g^2 + ideg + egfh + fhid. \end{aligned}$$

We will now state the result of this section, which for $n = 3$, $d = 2$ provides a more explicit description of the complement $\mathbb{C}[y_1, y_2, y_3]_3 \setminus \text{im}(\mathbf{A})$ than the one given by Theorem 3.3.

Proposition 4.1. *One has $\mathbb{C}[y_1, y_2, y_3]_3 \setminus \text{im}(\mathbf{A}) = \{S = 0\}$.*

Proof. We utilize canonical forms of ternary cubics. Namely, every nonzero ternary cubic is linearly equivalent to one of the following:

$$\begin{aligned} c_{1,t} &:= y_1^3 + y_2^3 + y_3^3 + ty_1y_2y_3, & t^3 \neq -27, \\ c_2 &:= y_1^3 + y_2^2y_3 & (\text{cuspidal cubic}), \\ c_3 &:= y_1^3 + y_1^2y_3 + y_2^2y_3 & (\text{nodal cubic}), \\ c_4 &:= y_1^2y_3 + y_2y_3^2, \\ c_5 &:= y_1^3 + y_1y_2y_3, \\ c_6 &:= y_1y_2y_3, \\ c_7 &:= y_1^2y_2 + y_1y_2^2, \\ c_8 &:= y_1^2y_2, \\ c_9 &:= y_1^3 \end{aligned}$$

(see, e.g., [K, p. 44]). Using formula (4.1) it is now easy to deduce

$$\{S = 0\} = \{0\} \cup O(c_{1,0}) \cup O(c_2) \cup O(c_4) \cup O(c_7) \cup O(c_8) \cup O(c_9),$$

where for a ternary cubic c we denote by $O(c)$ its GL_3 -orbit. In particular, we have $\{S = 0\} = \overline{O(c_{1,0})}$, which is the closure of the locus of ternary forms representable as the sum of the cubes of three linear forms (cf. [Ba, Theorems 2.1, 2.2] and [DK, Proposition 5.13.2]).

To see that $\text{im}(\mathbf{A})$ does not intersect the zero locus of S , we find the degree two component of the annihilator of each of the cubics $c_{1,0}, c_2, c_4, c_7, c_8, c_9$:

$$\begin{aligned} c_{1,0}^\perp \cap \mathbb{C}[x_1, x_2, x_3]_2 &= \langle x_1x_2, x_1x_3, x_2x_3 \rangle, \\ c_2^\perp \cap \mathbb{C}[x_1, x_2, x_3]_2 &= \langle x_1x_2, x_1x_3, x_3^2 \rangle, \\ c_4^\perp \cap \mathbb{C}[x_1, x_2, x_3]_2 &= \langle x_1^2 - x_2x_3, x_1x_2, x_2^2 \rangle, \\ c_7^\perp \cap \mathbb{C}[x_1, x_2, x_3]_2 &= \langle x_1^2 + x_2^2 - x_1x_2, x_1x_3, x_2x_3, x_3^2 \rangle, \\ c_8^\perp \cap \mathbb{C}[x_1, x_2, x_3]_2 &= \langle x_1x_3, x_2^2, x_2x_3, x_3^2 \rangle, \\ c_9^\perp \cap \mathbb{C}[x_1, x_2, x_3]_2 &= \langle x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2 \rangle. \end{aligned}$$

We thus see that for the cubics c_7, c_8, c_9 the corresponding annihilator components have dimension greater than 3 and that in the remaining situations they have zeroes away from the origin. It then follows that

$$\text{im}(\mathbf{A}) \subset \mathbb{C}[y_1, y_2, y_3]_3 \setminus \{S = 0\}.$$

In order to show that \mathbf{A} maps $(\mathbb{C}[x_1, x_2, x_3]_2^{\oplus 3})_{\text{Res}}$ onto $\mathbb{C}[y_1, y_2, y_3]_3 \setminus \{S = 0\}$, we need to prove that each of the cubics $c_{1,t}, c_3, c_5, c_6$ lies in $\text{im}(\mathbf{A})$, where $t \neq 0$, $t^3 \neq 216$ (notice that $c_{1,0}$ and $c_{1,\tau}$ with $\tau^3 = 216$ are linearly equivalent—see, e.g., [AIK, p. 603]). First of all, $c_{1,t}$, with $t \neq 0$, $t^3 \neq 216$, is proportional to the associated form of the nondegenerate cubic $c_{1,-18/t}$ and c_6 to the associated form

of the nondegenerate cubic $c_{1,0}$ (see, e.g., [AIK, Section 2.2]). Next, we calculate the degree two component of the annihilator of each of the cubics c_3, c_5 :

$$\begin{aligned} c_3^\perp \cap \mathbb{C}[x_1, x_2, x_3]_2 &= \langle x_1^2 - x_2^2 - 3x_1x_3, x_1x_2, x_3^2 \rangle, \\ c_5^\perp \cap \mathbb{C}[x_1, x_2, x_3]_2 &= \langle x_1^2 - 6x_2x_3, x_2^2, x_3^2 \rangle. \end{aligned}$$

This shows that c_3, c_5 lie in U_{Res} hence in $\text{im}(\mathbf{A})$.

The proof is now complete. \square

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